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Auxiliary tensor fields for $\mathrm{Sp}(2,\mathbb{R})$ self-duality

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ABSTRACT

The coset $\mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(1)$ is parametrized by two real scalar fields. We generalize the formalism of auxiliary tensor (bispinor) fields in $\mathrm{U}(1)$ self-dual nonlinear models of abelian gauge fields to the case of $\mathrm{Sp}(2,\mathbb{R})$ self-duality. In this new formulation, $\mathrm{Sp}(2,\mathbb{R})$ duality of the nonlinear scalar-gauge equations of motion is equivalent to an $\mathrm{Sp}(2,\mathbb{R})$ invariance of the auxiliary interaction. We derive this result in two different ways, aiming at its further application to supersymmetric theories. We also consider an extension to interactions with higher derivatives.

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1 Introduction

Noncompact $\mathrm{Sp}(2,\mathbb{R})$ duality arises in nonlinear electrodynamics interacting with dilaton and axion scalar fields which support a nonlinear realization of $\mathrm{Sp}(2,\mathbb{R})$ in the coset $\mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(1)$ [1]-[5]. The $\mathrm{Sp}(2,\mathbb{R})$ self-dual Lagrangian contains a specific interaction of the electromagnetic field F_{mn} and the coset scalar fields $S=(S_1,S_2)$ [3],

$$L^{\mathrm{sd}}(S,F)=L(S)+\tfrac{1}{4}S_1F^{mn}\tilde{F}_{mn}+\hat{L}(\sqrt{S_2}F_{mn})\,, \tag{1.1}$$

where $\hat{L}(F)=-\frac{1}{4}F^{mn}F_{mn}+\hat{L}^{\mathrm{int}}(F)$ is any Lagrangian of U(1) duality-invariant systems of nonlinear electrodynamics [1, 2], and $L(S)$ is a sigma-model-type Lagrangian for the coset scalar fields. Like in the U(1) duality case, the $\mathrm{Sp}(2,\mathbb{R})$ is respected by the full system of equations of motion following from (1.1) together with the Bianchi identity for $F^{mn}\sim(F_{\alpha\beta},\bar{F}_{\dot\alpha\dot\beta})$, while (1.1) on its own does not possess either $\mathrm{Sp}(2,\mathbb{R})$ or U(1) invariance. Coset-field-extended self-dual systems of abelian gauge fields are of interest as they naturally appear in extended supergravities (see, e.g., [6] and references therein).

The auxiliary-tensor formalism for U(1) self-duality [7]-[10] is based on extending the pure F^{mn} system by an auxiliary bispinor field $(V_{\alpha\beta},\bar{V}_{\dot\alpha\dot\beta})\sim V^{mn}$ which is not subject to any extra constraints like the Bianchi identity. In the extended formulation, U(1) self-duality is equivalent to manifest U(1) invariance of the nonlinear auxiliary interaction $E(V)$. Solving the algebraic equations for $V_{\alpha\beta}$ as $V_{\alpha\beta}=V_{\alpha\beta}(F)$, we regain the self-dual nonlinear Lagrangian $\hat{L}(F)$ in the standard representation. The main advantage of the formulation with auxiliary tensor fields is that it reduces the construction of the most general duality-invariant action for F^{mn} to listing all U(1) invariant auxiliary interactions $E(V)$. Later on, this approach was generalized to the $\mathrm{U}(N)$ case [11] and to self-dual systems of nonlinear $\mathcal{N}=1$ and $\mathcal{N}=2$ electrodynamics [12, 13, 14].

The natural next step is to include coset fields into the formalism of auxiliary (super)fields, first at the purely bosonic level and then, taking this as a prerequisite, to pass to the corresponding superfield extensions. In this paper we address the first part of this program. Namely, we generalize the formalism of auxiliary tensor fields to $\mathrm{Sp}(2,\mathbb{R})$ duality invariant theories.

In Section 2 we start with a review of the Gibbons-Rasheed construction of the axion-dilaton couplings in nonlinear electrodynamics. In Section 3 we introduce the auxiliary bispinor complex field $\hat{V}_{\alpha\beta}$ which transforms covariantly under the nonlinear realization (NR) of $\mathrm{Sp}(2,\mathbb{R})$. By construction, the basic Lagrangian obeys the Gaillard-Zumino representation and includes the $\mathrm{Sp}(2,\mathbb{R})$ invariant nonlinear interaction of the auxiliary fields $\hat{V}_{\alpha\beta}$. Solving the auxiliary-field equation in terms of the electromagnetic field $(F_{\alpha\beta},F_{\dot\alpha\dot\beta})$ we obtain the general self-dual $\mathrm{Sp}(2,\mathbb{R})$ Lagrangian in the standard representation. A more convenient construction of the auxiliary-field representation is based on a Legendre transformation. We also consider $\mathrm{Sp}(2,\mathbb{R})$ self-dual models with higher derivatives, employing covariant derivatives of the NR auxiliary fields. Section 4 describes an alternative formalism, which starts from auxiliary bispinor fields V transforming under a *linear* realization (LR) of $\mathrm{Sp}(2,\mathbb{R})$. The auxiliary interaction E in this formulation satisfies nonlinear constraints. The Legendre transformation allows one to linearize and solve these constraints, connecting E with the $\mathrm{Sp}(2,\mathbb{R})$ invariant interaction of auxiliary scalar fields. The two different choices of the auxiliary bispinor fields lead to equivalent F representations of self-dual Lagrangians.

2 Axion-dilaton coupling and nonlinear realization of $\mathrm{Sp}(2,\mathbb{R})$

The infinitesimal transformation of the group $\mathrm{Sp}(2,\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{R})$ (i.e. an element of the algebra $\mathrm{sp}(2,\mathbb{R})$) can be parametrized by the 2×2 matrix

$$\mathcal{B} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{2.1}$$

where a, b and c are real numbers. The nonlinear realization of $\mathrm{Sp}(2,\mathbb{R})$ is arranged as the transformations of the scalar field $S = S_1 + iS_2$ [3]

$$\delta S = b + 2aS - cS^2, \tag{2.2}$$

$$\delta S_1 = b + 2aS_1 - c(S_1^2 - S_2^2), \quad \delta S_2 = 2(a - cS_1)S_2\,, \tag{2.3}$$

where the real scalar fields S_1 and S_2 are connected with the axion A and dilaton, ϕ

$$S_1 = 2A, \quad S_2 = e^{-2\phi}. \tag{2.4}$$

The invariant Kähler σ model Lagrangian contains the Kähler metric $g_{S\bar{S}} = -\frac{1}{(S-\bar{S})^2}$,

$$L(S) = g_{S\bar{S}}\partial^m \bar{S}\partial_m S = -\frac{\partial^m \bar{S}\partial_m S}{(S-\bar{S})^2} = \frac{(\partial_m S_1)^2 + (\partial_m S_2)^2}{4\,S_2^2} = (e^{2\phi}\partial_m A)^2 + (\partial_m\phi)^2\,. \tag{2.5}$$

It is convenient to make the rescaling

$$L[S(A,\phi)] \;\to\; \frac{1}{\xi^2}L[S(\xi A, \xi\phi)]\,, \tag{2.6}$$

where ξ is a coupling constant of dimension -1.

The scalar equation of motion

$$E^0(S) := \frac{\Delta L_0(S)}{\Delta S} = \frac{1}{(S-\bar{S})^2}\left[\Box\bar{S} + \frac{2\partial^m\bar{S}\partial_m\bar{S}}{(S-\bar{S})}\right] = 0\,, \tag{2.7}$$

where $\Delta/\Delta S$ is the Euler-Lagrange derivative, transforms covariantly under the transformations (2.2):

$$\delta E^0(S) = -2(a - cS)\,E^0(S). \tag{2.8}$$

For the electromagnetic field strengths we will use both the bispinor and tensor representations

$$F_\alpha^\beta(A) \equiv \tfrac{1}{4}\partial_{\alpha\dot\beta}A^{\dot\beta\beta} - \tfrac{1}{4}\partial^{\dot\beta\beta}A_{\alpha\dot\beta} = \tfrac{1}{8}(\sigma^m\bar\sigma^n - \sigma^n\bar\sigma^m)_\alpha^\beta F_{mn}, \tag{2.9}$$

$$\varphi := F^{\alpha\beta}F_{\alpha\beta} = t + iz, \quad \bar\varphi = \bar{F}^2 = t - iz,$$

$$t := \tfrac{1}{4}F^{mn}F_{mn}, \quad z := \tfrac{1}{4}F^{mn}\tilde{F}_{mn}. \tag{2.10}$$

The general U(1) self-dual Lagrangian $\hat{L}(\varphi,\bar\varphi) = L'(t,z)$ satisfies the nonlinear differential condition [1, 2]

$$\mathrm{Im}\,[\varphi - 4\varphi(\hat{L}_\varphi)^2] = 0\,. \tag{2.11}$$

In this notation, the Gibbons-Rasheed (GR) Lagrangian (1.1) describing the interaction of scalar fields with the electromagnetic field in the nonlinear electrodynamics [3] is rewritten as

$$\begin{aligned} L^{\rm sd}(S,F) &= L(S) - \tfrac{i}{2}S_1\varphi + \tfrac{i}{2}S_1\bar\varphi + \hat L(\hat\varphi,\hat{\bar\varphi}) \\ &= L(S) + S_1 z + L'(S_2 t, S_2 z) := L(S) + \tilde L^{\rm sd}(S,F)\,. \end{aligned} \tag{2.12}$$

Here, $\hat L(\hat\varphi,\hat{\bar\varphi})$ is the same U(1) self-dual Lagrangian as before, but with the rescaled arguments,

$$\hat\varphi = S_2\varphi\,, \qquad \hat{\bar\varphi} = S_2\bar\varphi\,. \tag{2.13}$$

Evidently, $\hat L(\hat\varphi,\hat{\bar\varphi})$ satisfies the same condition as (2.11):

$$\mathrm{Im}\,[\hat\varphi - 4\,\hat\varphi(\hat L_{\hat\varphi})^2] = 0\,. \tag{2.14}$$

In what follows, we will need the explicit expression for the part of (2.12) linear in φ and $\bar\varphi$:

$$L^{\rm sd}_2(S,F) = \tilde L^{\rm sd}_2(S,F) = -\tfrac{i}{2}(\bar S\varphi - S\bar\varphi) = S_1 z - S_2 t\,, \tag{2.15}$$

where we took into account that

$$\hat L(\varphi,\bar\varphi) = -\tfrac{1}{2}(\varphi+\bar\varphi) + \hat L^{int}(\varphi,\bar\varphi)\,. \tag{2.16}$$

Now we consider the Bianchi identity for $F_{\alpha\beta}, \bar F_{\dot\alpha\dot\beta}$

$$B_{\alpha\dot\alpha} = \partial^{\dot\beta}_{\alpha}\bar F_{\dot\alpha\dot\beta} - \partial^{\beta}_{\dot\alpha}F_{\alpha\beta} = 0 \tag{2.17}$$

together with the nonlinear F-equation of motion

$$E_{\alpha\dot\alpha}(S,F) = \partial^{\dot\beta}_{\alpha}\bar P_{\dot\alpha\dot\beta} - \partial^{\beta}_{\dot\alpha}P_{\alpha\beta} = 0\,, \tag{2.18}$$

where the dual bispinor field $P_{\alpha\beta}$ and its conjugate $\bar P_{\dot\alpha\dot\beta}$ are defined by

$$P_{\alpha\beta}(S,F) = i\frac{\partial L^{\rm sd}}{\partial F^{\alpha\beta}} = (S_1 + 2iS_2\hat L_{\hat\varphi})\,F_{\alpha\beta} =: \tfrac{1}{8}(\sigma^m\bar\sigma^n - \sigma^n\bar\sigma^m)_{\alpha\beta}G_{mn}\,. \tag{2.19}$$

The whole dependence of the set of equations (2.17), (2.19) on the $\mathrm{Sp}(2,\mathbb{R})$ scalar fields S_1, S_2 is hidden in the tensors $P_{\alpha\beta}$ and $\bar P_{\dot\alpha\dot\beta}$.

The self-duality condition (2.14) guarantees the self-consistency of the linear $\mathrm{Sp}(2,\mathbb{R})$ transformations mixing $F_{\alpha\beta}$ with $P_{\alpha\beta}$

$$\delta P_{\alpha\beta} = aP_{\alpha\beta} + bF_{\alpha\beta}\,, \tag{2.20}$$

$$\delta F_{\alpha\beta} = cP_{\alpha\beta} - aF_{\alpha\beta} = -(a - cS_1 - 2icS_2\hat L_{\hat\varphi})\,F_{\alpha\beta}\,. \tag{2.21}$$

The pair of equations (2.17) and (2.18) is covariant under these $\mathrm{Sp}(2,\mathbb{R})$ duality transformation. Eqs. (2.20), (2.21) mean that the bispinors $F_{\alpha\beta}$ and $P_{\alpha\beta}$ form a linear $\mathrm{Sp}(2,\mathbb{R})$ doublet, so it is natural to use the notation $R^a := (P,F)$ and rewrite (2.20), (2.21) as $\delta R^a = \mathcal{B}^a_b R^b$, where the matrix $\mathcal{B}^a_b$ was defined in (2.1).

For further use we will also introduce the modified NR field strength and dual field strength

$$\hat{P}_{\alpha\beta} = i\frac{\partial\hat{L}}{\partial\hat{F}^{\alpha\beta}} = 2i\hat{F}_{\alpha\beta}\hat{L}_{\hat{\varphi}}, \quad \hat{F}_{\alpha\beta} = \sqrt{S_2}F_{\alpha\beta}\,, \tag{2.22}$$

in terms of which the self-duality condition (2.14) takes the conventional form [1]-[4]:

$$\mathrm{Im}(\hat{P}^2 + \hat{F}^2) = 0\,. \tag{2.23}$$

The modified quantities (2.22) transform nonlinearly under $\mathrm{Sp}(2,\mathbb{R})$. The relation between the NR representation (2.22) and the linearly transforming fields $P_{\alpha\beta}$ and $F_{\alpha\beta}$ can be written in the matrix form as

$$\hat{R} = \begin{pmatrix} \hat{P} \\ \hat{F} \end{pmatrix} = g\begin{pmatrix} P \\ F \end{pmatrix}, \quad R = \begin{pmatrix} P \\ F \end{pmatrix} = g^{-1}\begin{pmatrix} \hat{P} \\ \hat{F} \end{pmatrix}. \tag{2.24}$$

These relations contain the real coset matrix g and its inverse $g^{-1}(S)$:

$$g(S) = \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix} = \frac{1}{\sqrt{S_2}}\begin{pmatrix} 1 & -S_1 \\ 0 & S_2 \end{pmatrix}, \quad g^{-1}(S) = \frac{1}{\sqrt{S_2}}\begin{pmatrix} S_2 & S_1 \\ 0 & 1 \end{pmatrix}. \tag{2.25}$$

Transformations of the coset matrix have the following form:

$$\delta g = \Theta g - g\mathcal{B}, \tag{2.26}$$

$$\Theta = (\delta g)g^{-1} + g\mathcal{B}g^{-1} = \begin{pmatrix} 0 & -\rho \\ \rho & 0 \end{pmatrix} = -i\rho\tau_2\,, \tag{2.27}$$

where $\rho = cS_2$ is the induced parameter of the nonlinear realization, and τ_2 is the Pauli matrix. Thus the NR fields transform covariantly under the nonlinear realization of $\mathrm{Sp}(2,\mathbb{R})$

$$\delta\hat{P}_{\alpha\beta} = -\rho\hat{F}_{\alpha\beta}, \quad \delta\hat{F}_{\alpha\beta} = \rho\hat{P}_{\alpha\beta}\,. \tag{2.28}$$

The same transformations can be directly derived from the definition (2.22) with taking into account the compatibility constraint (2.14). For what follows, it is useful to explicitly write how $\hat{P}_{\alpha\beta}$ is expressed through $P_{\alpha\beta}$ and $F_{\alpha\beta}$

$$\hat{P}_{\alpha\beta} = \frac{1}{\sqrt{S_2}}(P_{\alpha\beta} - S_1F_{\alpha\beta}) \tag{2.29}$$

(the connection between $\hat{F}_{\alpha\beta}$ and $F_{\alpha\beta}$ was already given in (2.22)).

It is easy to construct, out of the coset fields S_1, S_2, the 2×2 matrix M_{ab} supporting a linear realization (LR) of $\mathrm{Sp}(2,\mathbb{R})$:

$$M(S) = g^Tg = \frac{1}{S_2}\begin{pmatrix} 1 & -S_1 \\ -S_1 & S_1^2 + S_2^2 \end{pmatrix}, \quad \det M = 1\,, \tag{2.30}$$

$$\delta M = -\mathcal{B}^TM - M\mathcal{B}\,.$$

One can alternatively write the coset Lagrangian (2.5) through this matrix

$$L(S) = -\tfrac{1}{4}\,\partial_mM_{ab}\partial^mM_{cd}\varepsilon^{ac}\varepsilon^{bd} \sim \mathrm{Tr}\,\partial_mM\partial^mM^{-1}\,. \tag{2.31}$$

The matrix of Kähler complex structure J^a_c also admits a simple expression through M

$$J^a_c(S) = M_{cb}(S)\varepsilon^{ba}, \quad J^a_c J^c_r = -\delta^a_r. \tag{2.32}$$

The standard U(1) duality relations are reproduced from the $\mathrm{Sp}(2,\mathbb{R})$ ones given above in the non-singular limit

$$S_1 = 0\,, \quad S_2 = 1\,. \tag{2.33}$$

In this limit, $g^a_b \to \delta^a_b\,,\ M_{ab} \to \delta_{ab}$ and $\mathrm{Sp}(2,\mathbb{R})$ is reduced to its $O(2) \sim U(1)$ subgroup with $a = 0, b = -c$, which preserves (2.33) and is just the standard U(1) duality group.

3 Nonlinear auxiliary fields in $\mathrm{Sp}(2,\mathbb{R})$ duality

3.1 $\mathrm{Sp}(2,\mathbb{R})$ duality as invariance of the auxiliary interaction

We introduce the following complex combinations of the NR bispinor fields $\hat{F}$ and $\hat{P}$ defined in (2.22):

$$\begin{aligned}
&\hat{V}_{\alpha\beta} = \tfrac{1}{2}(\hat{F} + i\hat{P})_{\alpha\beta}\,, \quad \delta\hat{V}_{\alpha\beta} = -i\rho\hat{V}_{\alpha\beta}\,,\\
&\hat{\bar{V}}_{\dot{\alpha}\dot{\beta}} = \tfrac{1}{2}(\hat{\bar{F}} - i\hat{\bar{P}})_{\dot{\alpha}\dot{\beta}}\,, \quad \delta\hat{\bar{V}}_{\dot{\alpha}\dot{\beta}} = i\rho\hat{\bar{V}}_{\dot{\alpha}\dot{\beta}}\,.
\end{aligned} \tag{3.1}$$

We will treat these fields as independent auxiliary tensor variables of the NR *representation* of the tensor formulation of the $\mathrm{Sp}(2,\mathbb{R})$ self-duality and *postulate* for $\hat{F}_{\alpha\beta}, \hat{P}_{\alpha\beta}$ just the NR transformation properties (2.28). The standard expression of $\hat{P}$ through the original fields $F_{\alpha\beta}, S_1, S_2$ as given in (2.22) will arise after eliminating $\hat{V}_{\alpha\beta}, \hat{\bar{V}}_{\dot{\alpha}\dot{\beta}}$ from the appropriate extended action by their equations of motion. In this extended formulation, we will also use the basic transformation of the NR electromagnetic field

$$\delta\hat{F}_{\alpha\beta} = i\rho(\hat{F} - 2\hat{V})_{\alpha\beta}, \quad \to \quad \delta(\hat{F} - \hat{V})_{\alpha\beta} = i\rho(\hat{F} - \hat{V})_{\alpha\beta}\,,$$

which is just the result of substituting $P_{\alpha\beta} = i(\hat{F}_{\alpha\beta} - 2V_{\alpha\beta})$ from the definition (3.1) into (2.28). The scalar combinations of the auxiliary fields and their transformation laws are given by

$$\hat{\nu} = \hat{V}^{\alpha\beta}\hat{V}_{\alpha\beta}, \quad \delta\hat{\nu} = -2i\rho\hat{\nu},$$

$$\hat{\bar{\nu}} = \hat{\bar{V}}^{\dot{\alpha}\dot{\beta}}\hat{\bar{V}}_{\dot{\alpha}\dot{\beta}}, \quad \delta\hat{\bar{\nu}} = 2i\rho\hat{\bar{\nu}}, \tag{3.2}$$

$$\hat{a} = \hat{\nu}\hat{\bar{\nu}}, \qquad \delta\hat{a} = 0\,. \tag{3.3}$$

By analogy with the auxiliary tensor field formulation of the U(1) duality [7]-[10], in constructing the extended action we start by defining the bilinear in F and $\hat{V}$ part of the interaction with scalar fields

$$\mathcal{L}_2(S, F, \hat{V}) = \tfrac{1}{2}(S_2 - iS_1)F^2 + \hat{V}^2 - 2\sqrt{S_2}(\hat{V}\cdot F) + \text{c.c.}\,. \tag{3.4}$$

By construction, the F derivative of this Lagrangian,

$$\frac{\partial\mathcal{L}_2}{\partial F^{\alpha\beta}} =: -iP_{\alpha\beta}(S, F, \hat{V}) = (S_2 - iS_1)F_{\alpha\beta} - 2\sqrt{S_2}\hat{V}_{\alpha\beta} = -i(\sqrt{S_2}\hat{P}_{\alpha\beta} + S_1F_{\alpha\beta}) \tag{3.5}$$

(where we once more used (3.1)), together with the field $F_{\alpha\beta}$, transform linearly, according to the transformation laws (2.20) and (2.21). This implies, in particular, that the modified equation of motion for the electromagnetic field calculated from $\mathcal{L}_2$ transform through the Bianchi identity (2.17) for $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ and vice versa, thus exhibiting the $\mathrm{Sp}(2,\mathbb{R})$ self-duality at this level. The complex bispinor field

$$V_{\alpha\beta}(S,F,\hat{V}) = \tfrac{1}{2}(F+iP)_{\alpha\beta} = \tfrac{1}{2}[(1-S_2+iS_1)F_{\alpha\beta} + 2\sqrt{S_2}\hat{V}_{\alpha\beta}] \tag{3.6}$$

also transforms linearly

$$\delta V_{\alpha\beta} = (a-ic)V_{\alpha\beta} + \tfrac{1}{2}(ib+ic-2a)F_{\alpha\beta}\,, \tag{3.7}$$
$$\delta F_{\alpha\beta} = -2icV_{\alpha\beta} + (ic-a)F_{\alpha\beta}\,. \tag{3.8}$$

Varying $\mathcal{L}_2$ with respect to $\hat{V}_{\alpha\beta}$ we obtain the $\mathrm{Sp}(2,\mathbb{R})$-covariant auxiliary equation

$$\hat{V}_{\alpha\beta} = \sqrt{S_2}F_{\alpha\beta} = \hat{F}_{\alpha\beta}\,. \tag{3.9}$$

Substituting this solution back into $\mathcal{L}_2$, we reproduce the bilinear interaction of the electromagnetic field with scalars $L_2^{\mathrm{sd}}(S,F)$ (eq. (2.15)).

The bilinear Lagrangian (3.4) admits the Gaillard-Zumino-type representation

$$\mathcal{L}_2 = -\tfrac{i}{2}(F\cdot P) + I_2 + \text{c.c.}, \tag{3.10}$$
$$I_2 = [\hat{V}^2 - \sqrt{S_2}(F\cdot\hat{V})] = -\tfrac{1}{4}\,R^bR^c\,M_{bc}(S), \tag{3.11}$$
$$R^1 = P, \quad R^2 = F\,, \tag{3.12}$$

where I_2 is the complex $\mathrm{Sp}(2,\mathbb{R})$ invariant, and $M_{bc}(S)$ is the LR matrix (2.30). This representation allows us to easily find the $\mathrm{Sp}(2,\mathbb{R})$ variation of the bilinear Lagrangian,

$$\delta\mathcal{L}_2 = \tfrac{i}{2}c(\bar{P}^2 - P^2) + \tfrac{i}{2}b(\bar{F}^2 - F^2)\,. \tag{3.13}$$

Now we are prepared to write the total nonlinear Lagrangian in the $(S,F,\hat{V})$ representation. It is a sum of $\mathcal{L}_2$ and the $\mathrm{Sp}(2,\mathbb{R})$ invariant terms

$$\mathcal{L}(S,F,\hat{V}) = L(S) + \mathcal{L}_2(S,F,\hat{V}) + \mathcal{E}(\hat{a})\,, \tag{3.14}$$
$$\frac{\partial\mathcal{L}}{\partial F^{\alpha\beta}} = \frac{\partial\mathcal{L}_2}{\partial F^{\alpha\beta}} = -iP_{\alpha\beta}(S,F,\hat{V})\,,$$

where $\hat{a}$ is the invariant quartic auxiliary variable defined in (3.3). Since the equations of motion for the electromagnetic field are not modified as compared to the $\mathcal{L}_2$ case, they still exhibit, together with the Bianchi identity, the $\mathrm{Sp}(2,\mathbb{R})$ covariance. The equation for the auxiliary field

$$(\hat{F} - \hat{V})_{\alpha\beta} = \hat{V}_{\alpha\beta}\hat{\bar{\nu}}\frac{d\mathcal{E}}{d\hat{a}} \tag{3.15}$$

is also manifestly $\mathrm{Sp}(2,\mathbb{R})$ covariant. It is analogous to the twisted self-duality constraints considered in [15, 16]. Note that the equation of motion for the coset fields (2.7) is modified by a non-zero source depending on the fields $\hat{V}, F$. It is easy to show that the $\mathrm{Sp}(2,\mathbb{R})$ covariance of (2.7) is not affected by this modification, like in the original GR framework.

It is instructive to rewrite (3.14) in the more detailed form

$$\mathcal{L}(S,F,\hat{V}) = L(S) - \tfrac{i}{2}S_1(\varphi-\bar{\varphi}) + \tfrac{1}{2}(\hat{\varphi}+\hat{\bar{\varphi}}) + (\hat{\nu}+\hat{\bar{\nu}}) - 2[(\hat{V}\cdot\hat{F}) + (\hat{\bar{V}}\cdot\hat{\bar{F}})] + \mathcal{E}(\hat{a})\,. \qquad (3.16)$$

The sum of the last four terms in (3.16) precisely coincides with the extended U(1) self-dual Lagrangian of nonlinear electrodynamics of refs. [7] - [10], up to the rescaling $F_{\alpha\beta} \to \hat{F}_{\alpha\beta} = \sqrt{S_2}F_{\alpha\beta}$. Hence, by reasoning of these papers, it should yield the most general self-dual Lagrangian $\hat{L}(\hat{\varphi},\bar{\varphi})$ upon eliminating the auxiliary fields $\hat{V}_{\alpha\beta}, \hat{\bar{V}}_{\dot{\alpha}\dot{\beta}}$ by their equations of motion. We conclude that (3.14), (3.16) indeed yields the auxiliary bispinor field extension of the general $\mathrm{Sp}(2,\mathbb{R})$ self-dual GR Lagrangian (2.12). Let us point out that the whole information about the given $\mathrm{Sp}(2,\mathbb{R})$ self-dual system is encoded in the $\mathrm{Sp}(2,\mathbb{R})$ invariant function $\mathcal{E}(\hat{a})$ which is not subject to any constraints. Given some bispinor field representation of the standard U(1) self-dual action, we can promote it to that defining an $\mathrm{Sp}(2,\mathbb{R})$ self-dual system just according to the recipe (3.16).

The auxiliary equation (3.15) is solved by

$$\hat{V}_{\alpha\beta} = \hat{F}_{\alpha\beta}\hat{G}(\hat{\varphi},\hat{\bar{\varphi}}), \quad \hat{G} = \frac{1}{1+\hat{\nu}\frac{d\mathcal{E}}{d\hat{a}}} = \frac{1}{2} - \frac{\partial\hat{L}}{\partial\hat{\varphi}}\,. \qquad (3.17)$$

By analogy with the U(1) case [10] we can use the perturbative expansion for $\mathcal{E}(\hat{a})$

$$\mathcal{E}(\hat{a}) = e_1\hat{a} + \tfrac{1}{2}e_2\hat{a}^2 + \ldots\,, \qquad (3.18)$$

where $e_1, e_2, \ldots$ are some constant coefficients. The corresponding perturbative solution for $\hat{L}$ reads

$$\begin{aligned}\hat{L}(\hat{\varphi},\hat{\bar{\varphi}}) &= -\tfrac{1}{2}(\hat{\varphi}+\hat{\bar{\varphi}}) + e_1\hat{\varphi}\hat{\bar{\varphi}} - e_1^2(\hat{\varphi}^2\hat{\bar{\varphi}} + \hat{\varphi}\hat{\bar{\varphi}}^2) + e_1^3(\hat{\varphi}^3\hat{\bar{\varphi}} + \hat{\varphi}\hat{\bar{\varphi}}^3)\\ &+(4e_1^3 + \tfrac{1}{2}e_2)\hat{\varphi}^2\hat{\bar{\varphi}}^2 - e_1^4(\hat{\varphi}^4\hat{\bar{\varphi}} + \hat{\varphi}\hat{\bar{\varphi}}^4) - (10e_1^4 + 2e_1e_2)(\hat{\varphi}^3\hat{\bar{\varphi}}^2 + \hat{\varphi}^2\hat{\bar{\varphi}}^3) + \ldots\,. \qquad (3.19)\end{aligned}$$

Like in the U(1) duality case, it is useful to define the intermediate (on-shell) representation for the self-dual Lagrangian by expressing (formally) the field $\hat{F}$ in terms of $\hat{V}$ from the algebraic equation (3.15):

$$\begin{aligned}F_{\alpha\beta} &\to F_{\alpha\beta}(\hat{V}) = \frac{1}{\sqrt{S_2}}(1+\hat{\bar{\nu}}\frac{d\mathcal{E}}{d\hat{a}})\hat{V}_{\alpha\beta}\,, \qquad (3.20)\\ \hat{P}_{\alpha\beta} &= i(\hat{F}-2\hat{V})_{\alpha\beta} \to \hat{P}_{\alpha\beta}(\hat{V}) = i[\hat{F}(\hat{V}) - 2\hat{V}]_{\alpha\beta}\,.\end{aligned}$$

This "on-shell" representation preserves the $\mathrm{Sp}(2,\mathbb{R})$ covariance

$$\delta[\hat{F}(\hat{V}) - \hat{V}]_{\alpha\beta} = i\rho[\hat{F}(\hat{V}) - \hat{V}]_{\alpha\beta}\,. \qquad (3.21)$$

The substitution (3.20) gives

$$\begin{aligned}\mathcal{L}_2(S,F,\hat{V}) &\to \tfrac{1}{2}(S_2 - iS_1)[F(\hat{V})\cdot F(\hat{V})] + \hat{V}^2 - 2\sqrt{S_2}[\hat{V}\cdot F(\hat{V})] + \text{c.c.}\\ &= -\frac{i}{2}[P(\hat{V})\cdot F(\hat{V})] - \hat{a}\frac{d\mathcal{E}}{d\hat{a}} + \text{c.c.}\,. \qquad (3.22)\end{aligned}$$

The same transform applied to the total Lagrangian (3.14) preserves the GZ form of the latter

$$\mathcal{L}(S,F,\hat{V}) \to \tilde{\mathcal{L}}(S,\hat{V}) = L(S) + \tfrac{i}{2}[\bar{P}(\hat{V})\cdot\bar{F}(\hat{V}) - P(\hat{V})\cdot F(\hat{V})] - 2\hat{a}\frac{d\mathcal{E}}{d\hat{a}} + \mathcal{E}(\hat{a}). \qquad (3.23)$$

Substituting here the solution $\hat{V}(\hat{F})$ (3.17), one recovers the F representation of the Lagrangian, i.e. (2.12).

Being applied to eq. (3.6), the change (3.20) yields

$$\begin{aligned} V_{\alpha\beta}(S,F,\hat{V}) \;\;\rightarrow\;\; & V_{\alpha\beta}(S,\hat{V}) = \tfrac{1}{2}[(1-S_2+iS_1)F_{\alpha\beta}(\hat{V}) + 2\sqrt{S_2}\hat{V}_{\alpha\beta}] \\ = \;\; & \left[\frac{1}{2\sqrt{S_2}}(1+\hat{\bar{\nu}}\frac{d\mathcal{E}}{d\hat{a}})(1-S_2+iS_1)+\sqrt{S_2}\right]\hat{V}_{\alpha\beta}\,. \end{aligned} \tag{3.24}$$

This establishes the relation between the LR and NR auxiliary fields (on the shell of the auxiliary equation (3.15)), which involves the scalar coset fields and the invariant interaction $\mathcal{E}$. The corresponding "on-shell" relation between the scalar combinations of the auxiliary fields reads

$$\nu(S,\hat{V}) = \left[\frac{1}{2\sqrt{S_2}}(1+\hat{\bar{\nu}}\frac{d\mathcal{E}}{d\hat{a}})(1-S_2+iS_1)+\sqrt{S_2}\right]^2\hat{\nu}. \tag{3.25}$$

3.2 Legendre transformation for the nonlinear auxiliary fields

The Legendre transformation for the auxiliary field formulation of the U(1) self-dual electrodynamics was discussed in [9, 10]. This transformation simplifies solving the auxiliary self-duality equation, which is the central step in deriving the conventional self-dual Lagrangian from the extended one.

To generalize this to the theory with the $\mathrm{Sp}(2,\mathbb{R})$ scalars, we introduce some new NR covariant scalar auxiliary fields $\hat{\mu}$ and $\hat{\bar{\mu}}$ with the transformation law

$$\delta\hat{\mu} = 2i\rho\hat{\mu}, \quad \delta\hat{\bar{\mu}} = -2i\rho\hat{\bar{\mu}} \tag{3.26}$$

and define the following generalized Lagrangian simultaneously involving two types of the auxiliary fields:

$$\begin{aligned} \mathcal{L}(S,F,\hat{V},\hat{\mu}) \;\; = \;\; & L(S) - \tfrac{i}{2}S_1S_2^{-1}\hat{\varphi} + \tfrac{i}{2}S_1S_2^{-1}\hat{\bar{\varphi}} + \hat{\nu} + \hat{\bar{\nu}} - 2\,[(\hat{V}\cdot\hat{F}) + (\hat{\bar{V}}\cdot\hat{\bar{F}})] + \tfrac{1}{2}(\hat{\varphi}+\hat{\bar{\varphi}}) \\ & + \hat{\nu}\hat{\mu} + \hat{\bar{\nu}}\hat{\bar{\mu}} + I(\hat{b})\,, \end{aligned} \tag{3.27}$$

where $I(\hat{b})$ is a function of the new invariant auxiliary variable $\hat{b} = |\hat{\mu}|^2$. This representation guarantees the $\mathrm{Sp}(2,\mathbb{R})$ covariance of the electromagnetic-scalar equations. Using the $\hat{V}$ equation of motion,

$$\hat{V}_{\alpha\beta} = \frac{1}{1+\hat{\mu}}\,\hat{F}_{\alpha\beta}\,, \tag{3.28}$$

we can eliminate the variable $\hat{V}_{\alpha\beta}$ from (3.27):

$$\begin{aligned} & \tilde{\mathcal{L}}(S,\varphi,\hat{\mu}) = L(S) + \left[-\frac{i}{2}S_1\varphi + \frac{S_2\varphi(\hat{\mu}-1)}{2(1+\hat{\mu})} + \text{c.c.}\right] + I(\hat{b}) \\ & = L(S) + \tfrac{i}{2}\left[(\bar{F}\cdot\bar{P}) - (F\cdot P)\right] + I(\hat{b})\,. \end{aligned} \tag{3.29}$$

In this case we obtain the simple expressions for the dual fields

$$P_{\alpha\beta} = i\frac{\partial\tilde{\mathcal{L}}}{\partial F^{\alpha\beta}} = \left[S_1 + \frac{iS_2(\hat{\mu}-1)}{(1+\hat{\mu})}\right]F_{\alpha\beta} = \sqrt{S_2}\hat{P}_{\alpha\beta} + \frac{S_1}{\sqrt{S_2}}\hat{F}_{\alpha\beta}\,, \tag{3.30}$$

whence

$$\hat{P}_{\alpha\beta} = \frac{i(\hat{\mu}-1)}{1+\hat{\mu}}\,\hat{F}_{\alpha\beta}\,. \tag{3.31}$$

These explicit expressions provide the correct transformation laws for the relevant quantities.

In fact, the representation (3.27) just defines the Legendre transformation of the Lagrangian (3.16). Indeed, varying (3.27) with respect to the auxiliary field $\hat{\mu}$ yields

$$\hat{\nu} = -\frac{\partial I(\hat{b})}{\partial\hat{\mu}}\,. \tag{3.32}$$

Using this equation, $\hat{\mu}$ and $\hat{b}$ can be expressed in terms of $\hat{\nu}, \hat{\bar{\nu}}$, assuming that $(\frac{dI}{d\hat{b}})^{-1}$ is not singular at $\hat{b}=0$. After the elimination of $\mu, \bar{\mu}$, the Lagrangian (3.27) takes the form of (3.16) with

$$\mathcal{E}(\hat{a}) = I(\hat{b}) - 2\hat{b}\frac{dI}{d\hat{b}}\,. \tag{3.33}$$

Then it is easy to show that

$$\hat{\mu} = \frac{\partial\mathcal{E}(\hat{a})}{\partial\hat{\nu}}\,. \tag{3.34}$$

Varying $\tilde{\mathcal{L}}(S,\hat{\varphi},\hat{\mu})$ (3.29) with respect to $\hat{\mu}$, we arrive at the equation for the auxiliary scalar variables

$$\hat{\varphi} = S_2\varphi = -(\hat{\bar{\mu}} + 2\hat{b} + \hat{b}\,\hat{\mu})\,\frac{dI}{d\hat{b}}\,. \tag{3.35}$$

It is analogous to the corresponding equation in the U(1) self-dual theory. Solving it for $\hat{\mu} = \hat{\mu}(\hat{\varphi}, \hat{\bar{\varphi}})$, we obtain the equation for determining the self-dual Lagrangian $\hat{L}(\hat{\varphi}, \hat{\bar{\varphi}})$

$$\frac{1}{1+\hat{\mu}} = \frac{1}{2} - \frac{\partial\hat{L}}{\partial\hat{\varphi}}\,. \tag{3.36}$$

By analogy with [9, 10], as a notorious example, we can obtain the exact Lagrangian for the Born-Infeld (BI) theory extended by the $\mathrm{Sp}(2,\mathbb{R})$ scalar fields, proceeding from the invariant interaction

$$I(\hat{b}) = \frac{2\hat{b}}{\hat{b}-1}\,. \tag{3.37}$$

In this case, one can find the complete solution to the equation (3.35)

$$\hat{\mu} \;=\; \frac{Q-1-\frac{1}{2}(\hat{\varphi}-\hat{\bar{\varphi}})}{Q+1+\frac{1}{2}(\hat{\varphi}-\hat{\bar{\varphi}})}, \tag{3.38}$$

$$Q(\hat{\varphi}, \hat{\bar{\varphi}}) \;=\; \sqrt{1+(\hat{\varphi}+\hat{\bar{\varphi}})+\tfrac{1}{4}(\hat{\varphi}-\hat{\bar{\varphi}})^2} = Q(S_2\varphi, S_2\bar{\varphi}) = 1 - L_{BI}(S_2\varphi, S_2\bar{\varphi}). \tag{3.39}$$

Taking $L_{BI}(S_2\varphi, S_2\bar{\varphi})$ as $\hat{L}(\hat{\varphi}, \hat{\bar{\varphi}})$ in eq. (2.12), we recover the standard GR coupling of the scalar fields in the BI theory [3]. Note that the derivation of the BI Lagrangian in the $\hat{b}$ representation is much easier than in the original $\hat{a}$ formulation.

All other examples of the U(1) self-dual systems in which eq. (3.35) has a closed solution can also be generalized to the $\mathrm{Sp}(2,\mathbb{R})$ case. For instance, the invariant auxiliary interaction

$$I(\hat{b}) = 2\ln(1-\hat{b}) \tag{3.40}$$

yields the cubic equation for $\hat{\mu}(\hat{\varphi})$ and gives us the exact expression for $\hat{L}(\hat{\varphi},\bar{\hat{\varphi}})$, like in the analogous U(1) model considered in [10].

The scalar coupling in the so called "simplest interaction model" [7, 15, 10] corresponds to the choice $I_{SI} = -2\hat{b}$.

3.3 $\mathrm{Sp}(2,\mathbb{R})$ duality and higher derivatives

The self-dual theories with higher derivatives in the standard setting were analyzed in [15, 17]. The formulation through auxiliary tensor fields for such theories was worked out in [10]. As was shown there, any U(1) self-dual theory with higher derivatives is generated by the appropriate U(1) invariant auxiliary interaction involving space-time derivatives of the auxiliary bispinor fields .

The transformation parameter $\rho = cS_2$ in the NR representation (3.1) depends on the scalar field, so in the case under consideration we need to properly $\mathrm{Sp}(2,\mathbb{R})$ covariantize the space-time derivatives of the auxiliary fields involved. This can be done following the standard routine of the nonlinear realizations [18].

First, we construct the 2×2 matrix Cartan 1-forms pertinent to the nonlinear realization of $\mathrm{Sp}(2,\mathbb{R})$ in the symmetric coset space $\mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(1)$ we deal with:

$$dgg^{-1} = \Gamma + D, \tag{3.41}$$

$$\Gamma = dx^m\Gamma_m = \tfrac{1}{2}dgg^{-1} - \tfrac{1}{2}g^{-1T}dg^T = \begin{pmatrix} 0 & -\zeta \\ \zeta & 0 \end{pmatrix}, \tag{3.42}$$

$$D = dx^m D_m = \tfrac{1}{2}dgg^{-1} + \tfrac{1}{2}g^{-1T}dg^T = \begin{pmatrix} p & q \\ q & -p \end{pmatrix}. \tag{3.43}$$

Here, g is the coset matrix (2.25) and $g^{-1},\ g^T$ are the corresponding inverse and transposed matrices. The 1-form $\Gamma = -i\tau_2 dx^m\zeta_m$ contains the induced connection $\zeta_m(S)$ which defines the covariant derivatives of fields having the standard transformation properties, i.e. transforming with the induced U(1) parameter ρ, while D specifies the NR-covariant derivative $D_m(S)$ of the coset fields [1]. These objects have the following transformation rules

$$\delta\Gamma = d\Theta + [\Theta,\Gamma]\,, \quad \delta D = [\Theta, D]. \tag{3.44}$$

The connection 1-form reads

$$\zeta = dx^m\zeta_m = \tfrac{1}{2}(g^1_1)^2 dS_1\,, \quad \zeta_m = \frac{1}{2S_2}\partial_m S_1\,, \qquad \delta\zeta_m = \partial_m\rho\,. \tag{3.45}$$

The explicit expressions for the component 1-forms collected in the matrix (3.43) are as follows

$$p = dx^m p_m = dg^1_1 g^2_2 = -\frac{1}{2S_2}\,dS_2, \quad q = dx^m q_m = -\tfrac{1}{2}(g^1_1)^2 dS_1 = -\frac{1}{2S_2}\,dS_1, \tag{3.46}$$

$$\delta p_m = -2\rho q_m, \quad \delta q_m = 2\rho p_m, \quad \delta(p_m + iq_m) = 2i\rho(p_m + iq_m). \tag{3.47}$$

[1] The Lagrangian (2.5) is just the square of these covariant derivatives, $L(S) = \frac{1}{2}\mathrm{Tr}\, D_m(S)D^m(S)$.

Now we are ready to define the covariant derivatives of the NR auxiliary fields $\hat{V}$ and $\hat{\bar{V}}$ (we suppress their Lorentz indices)

$$\nabla_m \hat{V} = (\bar{\sigma}_m)^{\dot{\rho}\rho} \nabla_{\rho\dot{\rho}} \hat{V} = (\partial_m + i\zeta_m)\hat{V}, \quad \delta\nabla_m \hat{V} = -i\rho \nabla_m \hat{V}, \tag{3.48}$$

$$\nabla_m \hat{\bar{V}} = (\bar{\sigma}_m)^{\dot{\rho}\rho} \nabla_{\rho\dot{\rho}} \hat{\bar{V}} = (\partial_m - i\zeta_m)\hat{\bar{V}}, \quad \delta\nabla_m \hat{\bar{V}} = i\rho \nabla_m \hat{\bar{V}}. \tag{3.49}$$

The corresponding $\mathrm{Sp}(2,\mathbb{R})$ invariant auxiliary interaction can be constructed by analogy with the U(1) self-dual theory [10]. We should add to the standard bilinear interaction $\mathcal{L}_2(S, F, \hat{V})$, eq. (3.4), the general $\mathrm{Sp}(2,\mathbb{R})$ invariant interaction involving the covariant derivatives of the scalar coset fields and the NR auxiliary fields:

$$\mathcal{E}^K_{der}(\hat{V}, p_m(S), q_m(S), \nabla_m \hat{V}, \ldots, \nabla_{m_1}\nabla_{m_2}\cdots\nabla_{m_k}\hat{V}, \ldots)\,. \tag{3.50}$$

The $\mathrm{Sp}(2,\mathbb{R})$-covariant local equations of motion for the auxiliary fields in this case contain the Euler-Lagrange derivative

$$(\hat{V} - \hat{F})_{\alpha\beta} + \frac{1}{2}\frac{\Delta\hat{\mathcal{E}}^K_{der}}{\Delta\hat{V}^{\alpha\beta}} = 0\,. \tag{3.51}$$

Solving this equation (e.g., by recursions), we finally obtain the $\mathrm{Sp}(2,\mathbb{R})$ self-dual Lagrangian in the initial (F, S) representation.

The auxiliary interaction $\mathcal{E}^K_{der}$ specifying the $\mathrm{Sp}(2,\mathbb{R})$ self-dual models with higher derivatives involves new dimensionful constants, starting from the coupling constant c of dimension -2, as well as additional dimensionless coupling constants. Examples of interaction with two derivatives are provided by the terms

$$\sim \nabla^{\dot{\beta}}_{\beta}\hat{V}^{\alpha\beta}\nabla^{\dot{\xi}}_{\alpha}\hat{\bar{V}}_{\dot{\beta}\dot{\xi}}, \qquad \sim \nabla^m \hat{\nu}\nabla_m \hat{\bar{\nu}}, \ldots\,. \tag{3.52}$$

Non-standard terms with higher derivatives can be generated by the invariant combinations of the scalar and auxiliary fields, e.g.,

$$R_m = (p_m + iq_m)\hat{V}^2\,. \tag{3.53}$$

Note that terms with higher derivatives now also appear in the scalar S_1, S_2 equations.

4 Alternative auxiliary-field formulation of $\mathrm{Sp}(2,\mathbb{R})$ theory

In the previous section we started from the renowned GR action (1.1), (2.12) and picked up the nonlinearly transforming bispinor auxiliary fields $\hat{V}_{\alpha\beta}$ so as to construct the natural generalization of the extended formulation of the U(1) self-dual electrodynamics to the case of the $\mathrm{Sp}(2,\mathbb{R})$ self-dual systems with the coset scalar fields. In this section we present an alternative construction which starts just from the extended U(1) formulation and produces the GR action as an output. Its basic distinguishing feature is that it starts with the *linear* realization of $\mathrm{Sp}(2,\mathbb{R})$ on the set $(V_{\alpha\beta}, F_{\alpha\beta})$.

4.1 λ parametrization of the Sp$(2,\mathbb{R})$ coset

In the alternative construction it will be convenient to use another parametrization of the coset of Sp$(2,\mathbb{R})$, this time by the complex scalar field $\lambda(x)$:

$$\bar{S}(\lambda) = \frac{i(\lambda-1)}{\lambda+1}, \qquad \lambda(S) = \frac{1-i\bar{S}}{1+i\bar{S}}, \tag{4.1}$$

$$S_1(\lambda) = \frac{i(\lambda-\bar{\lambda})}{(1+\lambda)(1+\bar{\lambda})}, \qquad S_2(\lambda) = \frac{1-\lambda\bar{\lambda}}{(1+\lambda)(1+\bar{\lambda})}, \tag{4.2}$$

and use the alternative set of the group parameters

$$\alpha = -a - \tfrac{i}{2}(b+c), \quad \bar{\alpha} = -a + \tfrac{i}{2}(b+c), \quad \gamma = \tfrac{1}{2}(c-b), \tag{4.3}$$

$$a = -\tfrac{1}{2}(\alpha+\bar{\alpha}), \quad b = \tfrac{i}{2}(\alpha-\bar{\alpha}) - \gamma, \quad c = \tfrac{i}{2}(\alpha-\bar{\alpha}) + \gamma. \tag{4.4}$$

The new coset field has a simple transformation law

$$\delta\lambda = \delta\frac{2}{1+i\bar{S}} = \alpha + 2i\gamma\lambda - \bar{\alpha}\lambda^2\,, \tag{4.5}$$

which resembles the $\mathbb{CP}_1$ realization of the group SU(2), the only difference being the sign of the last term in (4.5).

The scalar Lagrangian (2.5) can be rewritten in terms of λ as

$$L(S) = L'(\lambda) = \frac{\partial^m\bar{\lambda}\partial_m\lambda}{(1-\lambda\bar{\lambda})^2}\,. \tag{4.6}$$

The relevant coset element can be represented by the Hermitian matrix

$$G = \frac{1}{\sqrt{1-\lambda\bar{\lambda}}}\begin{pmatrix} 1 & -\lambda \\ -\bar{\lambda} & 1 \end{pmatrix}, \tag{4.7}$$

with the transformation law

$$\delta G = i\tau_3\tilde{\gamma}G - G\Lambda = -\Lambda^\dagger G - i\tilde{\gamma}G\tau_3\,, \quad \tilde{\gamma}(\lambda) := \gamma + \tfrac{1}{2i}(\alpha\bar{\lambda} - \bar{\alpha}\lambda)\,, \tag{4.8}$$

where $\tilde{\gamma}(\lambda)$ is the corresponding induced parameter, τ_3 is the Pauli matrix and

$$\Lambda = \begin{pmatrix} i\gamma & \alpha \\ \bar{\alpha} & -i\gamma \end{pmatrix}. \tag{4.9}$$

The previously considered Sp$(2,\mathbb{R})$ spinors $R^a = (P,F)$ linearly transforming with the matrix $\mathcal{B}$ are related to the spinors T^a transforming with the new matrix Λ as

$$T = AR = A\begin{pmatrix} P \\ F \end{pmatrix}, \quad R = A^{-1}T, \quad \delta T = \Lambda T\,, \tag{4.10}$$

where

$$\Lambda = A\mathcal{B}A^{-1} = A\mathcal{B}A^\dagger\,, \quad \Lambda^\dagger = A\mathcal{B}^TA^\dagger\,, \tag{4.11}$$

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}, \quad A^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = A^\dagger\,.$$

The Cartan form in this representation, dGG^{-1}, contains the corresponding connection and covariant derivatives by analogy with (3.41):

$$dGG^{-1} = \frac{1}{2(1-\lambda\bar{\lambda})}\left(\begin{array}{cc} \lambda d\bar{\lambda} - \bar{\lambda}d\lambda & -2d\lambda \\ -2d\bar{\lambda} & -\lambda d\bar{\lambda} + \bar{\lambda}d\lambda \end{array}\right). \tag{4.12}$$

The off-diagonal elements in (4.12) are just the covariant differentials $D\lambda = dx^m D_m\lambda$ and $D\bar{\lambda} = dx^m D_m\bar{\lambda}$, $\delta D\lambda = 2i\tilde{\gamma}D\lambda$ (the Lagrangian (4.6) is bilinear in the covariant derivatives $D_m\lambda, D_m\bar{\lambda}$), while the diagonal element is the U(1) connection

$$\mathcal{A} = \mathcal{A}_m dx^m = \frac{1}{2i}\frac{\bar{\lambda}d\lambda - \lambda d\bar{\lambda}}{1-\lambda\bar{\lambda}}, \quad \delta\mathcal{A} = d\tilde{\gamma}\,. \tag{4.13}$$

It defines the covariant derivative of some field with the standard transformation law under the nonlinear realization considered

$$D_m\psi = (\partial_m - iq\mathcal{A}_m)\psi\,, \quad \delta\psi = iq\,\tilde{\gamma}\psi\,, \;\; \delta D_m\psi = iq\,\tilde{\gamma}D_m\psi\,, \tag{4.14}$$

where q is the U(1) charge of ψ.

The real symmetric LR matrix defined in (2.30) is related to G^2

$$M(\lambda) = A^{-1}G^2A = g^T(\lambda)g(\lambda), \quad \delta M(\lambda) = -\mathcal{B}^T M(\lambda) - M(\lambda)\mathcal{B}\,, \tag{4.15}$$

where $g(\lambda) := g[S(\lambda)]$ is the λ representation of the real coset matrix (2.25).

Two alternative $\mathrm{Sp}(2,\mathbb{R})$ cosets are connected by the intertwining matrix

$$\mathrm{Int} := GAg^{-1}, \quad \delta(\mathrm{Int}) = i\tilde{\gamma}\tau_3\,\mathrm{Int} + i\mathrm{Int}\,\tau_2\rho. \tag{4.16}$$

4.2 Auxiliary fields

The starting point of the alternative construction of the extended Lagrangian for the $\mathrm{Sp}(2,\mathbb{R})$ self-duality is just the U(1) Lagrangian of refs. [7] - [10]:

$$\mathcal{L}(F,V) = \tfrac{1}{2}(\varphi + \bar{\varphi}) + \nu + \bar{\nu} - 2[(V\cdot F) + (\bar{V}\cdot\bar{F})] + \mathcal{E}(a)\,, \quad a = \nu\bar{\nu}\,. \tag{4.17}$$

It is easy to check that the equations of motion and Bianchi identities for $F_{\alpha\beta}$,

$$\partial^\beta_{\dot{\alpha}}(F_{\alpha\beta} - 2V_{\alpha\beta}) + \text{c.c.} = 0\,, \tag{4.18}$$

$$\partial^\beta_{\dot{\alpha}}\,F_{\alpha\beta} - \text{c.c.} = 0\,, \tag{4.19}$$

are covariant under the $\mathrm{Sp}(2,\mathbb{R})$ rotations

$$\delta V_{\alpha\beta} = -i\gamma\,V_{\alpha\beta} + \bar{\alpha}\,(F-V)_{\alpha\beta}\,, \quad \delta(F-V)_{\alpha\beta} = i\gamma\,(F-V)_{\alpha\beta} + \alpha\,V_{\alpha\beta}\,, \tag{4.20}$$

or

$$\delta F_{\alpha\beta} = i\gamma(F_{\alpha\beta} - 2V_{\alpha\beta}) + \alpha V_{\alpha\beta} + \bar{\alpha}(F_{\alpha\beta} - V_{\alpha\beta})\,. \tag{4.21}$$

These linear transformations coincide with the transformations (3.7), (3.8) in the S_1, S_2 basis. At the same time, the algebraic equation

$$F_{\alpha\beta} - V_{\alpha\beta} = V_{\alpha\beta}\,\mathcal{E}_\nu\,, \tag{4.22}$$

is evidently not covariant. The question is how to modify eq. (4.22) in order to make it also $\mathrm{Sp}(2,\mathbb{R})$ covariant.

In the free case, with $\mathcal{E}(\nu,\bar\nu) = 0$, this modification is rather simple:

$$(F-V)_{\alpha\beta} = 0 \;\Rightarrow\; F_{\alpha\beta} - (1+\lambda)V_{\alpha\beta} = 0\,. \tag{4.23}$$

Using the transformation law of λ (4.5), it is easy to show that

$$\delta\left[F_{\alpha\beta} - (1+\lambda)V_{\alpha\beta}\right] = (i\gamma - \bar\alpha\lambda)\left[F_{\alpha\beta} - (1+\lambda)V_{\alpha\beta}\right], \tag{4.24}$$

which implies the covariance of (4.23).

As the next step, the following generalization of (4.23) naturally occurs:

$$F_{\alpha\beta} - (1+\lambda)V_{\alpha\beta} = V_{\alpha\beta}\,E_\nu\,, \qquad E = E(\nu,\bar\nu,\lambda,\bar\lambda)\,. \tag{4.25}$$

The function $E(\nu,\bar\nu,\lambda,\bar\lambda)$ is assumed to become the previous $E(\nu,\bar\nu) = \mathcal{E}(a)$ in the limit $\lambda = 0$ yielding the Lagrangian (4.17) with the residual U(1) duality group with the parameter γ. Eq. (4.25), together with the dynamical equation (4.18), can be obtained from the Lagrangian

$$\mathcal{L}'(\lambda,F,V) = L'(\lambda) + (1+\lambda)\nu + (1+\bar\lambda)\bar\nu - 2[(V\cdot F) + (\bar V\cdot\bar F)] + \tfrac{1}{2}(\varphi+\bar\varphi) + E(\nu,\bar\nu,\lambda\bar\lambda)\,. \tag{4.26}$$

For $E = 0$, using the algebraic equation (4.23), we obtain the following λ-modified free $(F,\bar F)$ Lagrangian:

$$L_2^{\rm sd}(\lambda,F) = \frac{1}{2}\,\frac{\lambda-1}{\lambda+1}\,\varphi + \frac{1}{2}\,\frac{\bar\lambda-1}{\bar\lambda+1}\,\bar\varphi\,. \tag{4.27}$$

With taking into account the relations (4.1), it coincides with the bilinear part (2.15) of the GR Lagrangian in the (S_1,S_2) parametrization.

In what follows, it will be convenient to deal with the modified interaction function

$$\hat E = E + \lambda\nu + \bar\lambda\bar\nu\,, \tag{4.28}$$

that corresponds to transferring the λ-dependent terms in the l.h.s. of (4.25) to its r.h.s. We will fix the λ-dependence of the interaction E (or $\hat E$) from the requirement of compatibility of the $\mathrm{Sp}(2,\mathbb{R})$ variations of the left- and right-hand sides of eq. (4.25). Using the transformation laws (4.20), we find that, on the shell of the auxiliary equation (4.25),

$$\delta\nu = -2i\gamma\,\nu + 2\bar\alpha\,\nu\hat E_\nu\,, \quad \delta\bar\nu = 2i\gamma\,\bar\nu + 2\alpha\,\bar\nu\hat E_{\bar\nu}\,. \tag{4.29}$$

Then, taking into account this transformation law together with (4.20), (4.5) and, once again, (4.25), we find the variation of the r.h.s. of (4.25) and compare it with that of the l.h.s., i.e., with (4.24). We find the following conditions[2] on the function $\hat E$

$$\nu\hat E_\nu - \bar\nu\hat E_{\bar\nu} - \lambda\hat E_\lambda + \bar\lambda\hat E_{\bar\lambda} = 0\,, \tag{4.30}$$
$$\nu(\hat E_\nu)^2 - \bar\nu + D_{\bar\lambda}\hat E = 0\,, \tag{4.31}$$
$$\bar\nu(\hat E_{\bar\nu})^2 - \nu + D_\lambda\hat E = 0\,, \tag{4.32}$$

[2] Actually, the original conditions are the vanishing of the ν and $\bar\nu$ derivatives of the equations below, but we assume that the integration constants (which depend only on λ and $\bar\lambda$) can be put equal to zero without loss of generality.

where

$$D_{\bar\lambda} = \partial_{\bar\lambda} - \lambda^2\partial_\lambda\,, \quad D_\lambda = \partial_\lambda - \bar\lambda^2\partial_{\bar\lambda}\,, \quad [D_{\bar\lambda}, D_\lambda] = 2(\lambda\partial_\lambda - \bar\lambda\partial_{\bar\lambda})\,. \tag{4.33}$$

Eq. (4.30) is just the condition of the U(1) invariance of the generalized function $\hat E(\nu, \bar\nu, \lambda, \bar\lambda)$. The mutually conjugated eqs. (4.31) and (4.32) are new. One can check that the same system of equations arises from the requirement that the transformations (4.29) have the correct $sp(2, \mathbb{R})$ closure.

Repeatedly using the constraints (4.31) and (4.32), one finds that

$$\delta \hat E_\nu = \alpha - \bar\alpha(\hat E_\nu)^2\,, \quad \delta \hat E_{\bar\nu} = \bar\alpha - \alpha(\hat E_{\bar\nu})^2\,, \tag{4.34}$$

and

$$\delta \hat E = \alpha\left[\nu + \bar\nu(\hat E_{\bar\nu})^2\right] + \bar\alpha\left[\bar\nu + \nu(\hat E_\nu)^2\right]. \tag{4.35}$$

An interesting peculiarity is that the transformation laws of $\hat E_\nu$ and $\hat E_{\bar\nu}$ exactly mimic those of λ and $\bar\lambda$. We also observe that $\hat E$ *is not invariant* under the coset $\mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(1)$ transformations, while it is still invariant under their U(1) closure. Surprisingly, we can construct such $\mathrm{Sp}(2,\mathbb{R})$ invariant from the two independent U(1) invariants

$$H(\nu, \bar\nu, \lambda, \bar\lambda) = \hat E - (\nu \hat E_\nu + \bar\nu \hat E_{\bar\nu}) = E - (\nu E_\nu + \bar\nu E_{\bar\nu})\,, \quad \delta H = 0\,. \tag{4.36}$$

A similar object already appeared in [9], when performing the Legendre transformation from the variables $\nu, \bar\nu$ to $\mu, \bar\mu$ (recall also Subsection 3.2).

At this step, we deal with the bispinor field extended Lagrangian (4.26), which is reduced to the extended Lagrangian (4.17) of the U(1) duality systems in the limit $\lambda = 0$ and exhibits $\mathrm{Sp}(2,\mathbb{R})$ duality under the constraints (4.30) - (4.32) on the interaction function $\hat E(\nu, \bar\nu, \lambda, \bar\lambda)$. The question is how to solve these constraints via some unconstrained "prepotential" which would be analogous to $\mathcal{E}(\hat a)$ of Subsection 3.1. While for the time being we do not know how to achieve this, the problem is radically simplified in the μ representation obtained by Legendre transformation of (4.26), with the $\mathrm{Sp}(2,\mathbb{R})$ invariant interaction H, eq. (4.36), instead of $\hat E$. In this representation, the constraints are *linearized*.

Let us define

$$\mu := \hat E_\nu\,, \quad \bar\mu = \hat E_{\bar\nu}\,. \tag{4.37}$$

In the μ representation:

$$\nu = -H_\mu\,, \quad \bar\nu = -H_{\bar\mu}, \quad H = H(\mu, \bar\mu, \lambda, \bar\lambda)\,. \tag{4.38}$$

The corresponding Legendre-transformed Lagrangian can be derived from

$$\mathcal{L}'(\lambda, F, V, \mu) = L'(\lambda) + \nu + \bar\nu - 2[(V\cdot F) + (\bar V \cdot \bar F)] + \tfrac{1}{2}(\varphi + \bar\varphi) + \nu\mu + \bar\nu\bar\mu + H(\mu, \bar\mu, \lambda\bar\lambda)\,. \tag{4.39}$$

Varying it with respect to $\mu, \bar\mu$ yields just eqs. (4.38), which express $\mu, \bar\mu$ in terms of $\nu, \bar\nu, \lambda, \bar\lambda$, taking us back to (4.26). On the other hand, eliminating $V_{\alpha\beta}$, we obtain an analog of the (F, μ) Lagrangian (3.29).

It is easy to show that the $\mathrm{Sp}(2,\mathbb{R})$ invariance conditions (4.31), (4.32) are indeed linearized in this μ-representation

$$D_\lambda H + (\partial_\mu - \bar\mu^2\partial_{\bar\mu})H = 0\,, \quad D_{\bar\lambda} H + (\partial_{\bar\mu} - \mu^2\partial_\mu)H = 0\,, \tag{4.40}$$

while (4.30) takes the form

$$\mu H_\mu - \bar\mu H_{\bar\mu} + \lambda H_\lambda - \bar\lambda H_{\bar\lambda} = 0\,. \tag{4.41}$$

Using the transformation properties

$$\delta\mu = \alpha - \bar\alpha(\mu)^2\,, \quad \delta\bar\mu = \bar\alpha - \alpha(\bar\mu)^2\,, \tag{4.42}$$

it is easy to check that the quantities

$$\tilde\mu = \frac{\mu - \lambda}{1 - \mu\bar\lambda}\,, \qquad \bar{\tilde\mu} = \frac{\bar\mu - \bar\lambda}{1 - \bar\mu\lambda}\,, \tag{4.43}$$

possess the standard nonlinear realization transformation law

$$\delta\tilde\mu = 2i\tilde\gamma\,\tilde\mu\,, \quad \delta\bar{\tilde\mu} = -2i\tilde\gamma\,\bar{\tilde\mu}\,. \tag{4.44}$$

Then we define the $\mathrm{Sp}(2,\mathbb{R})$ invariant real argument

$$b := \tilde\mu\,\bar{\tilde\mu}\,, \tag{4.45}$$

and check that

$$D_\lambda b + (\partial_\mu - \bar\mu^2\partial_{\bar\mu})b = 0\,, \quad \text{and c.c.}\,. \tag{4.46}$$

This immediately implies that the solution of the constraints (4.40) is

$$H(\mu,\bar\mu,\lambda,\bar\lambda) = I(b)\,. \tag{4.47}$$

Another way to achieve the same result is to make the change of variables $(\mu,\bar\mu,\lambda,\bar\lambda) \to (\tilde\mu,\bar{\tilde\mu},\lambda,\bar\lambda)$, $H(\mu,\bar\mu,\lambda,\bar\lambda) = \tilde H(\tilde\mu,\bar{\tilde\mu},\lambda,\bar\lambda)$, in the constraints (4.40) and (4.41). Using the relations

$$\begin{aligned} \mu &= \frac{\tilde\mu + \lambda}{1 + \tilde\mu\bar\lambda}\,, \qquad 1 + \mu = \frac{1 + \lambda + \tilde\mu(1+\bar\lambda)}{1 + \tilde\mu\bar\lambda}\,, \\ H_\mu &= \tilde H_{\tilde\mu}\,\frac{1 - \lambda\bar\lambda}{(1 - \mu\bar\lambda)^2} = \tilde H_{\tilde\mu}\,\frac{(1 + \tilde\mu\bar\lambda)^2}{1 - \lambda\bar\lambda}\,, \end{aligned} \tag{4.48}$$

one can check that in the new basis the constraints take the form

$$D_\lambda\tilde H + \bar\lambda(\tilde\mu\tilde H_{\tilde\mu} - \bar{\tilde\mu}\tilde H_{\bar{\tilde\mu}}) = 0\,, \quad D_{\bar\lambda}\tilde H - \lambda(\tilde\mu\tilde H_{\tilde\mu} - \bar{\tilde\mu}\tilde H_{\bar{\tilde\mu}}) = 0\,, \tag{4.49}$$

$$\tilde\mu\tilde H_{\tilde\mu} - \bar{\tilde\mu}\tilde H_{\bar{\tilde\mu}} + \lambda\tilde H_\lambda - \bar\lambda\tilde H_{\bar\lambda} = 0\,. \tag{4.50}$$

They are equivalent to the set

$$\tilde H_\lambda = \tilde H_{\bar\lambda} = 0\,, \quad \tilde\mu\tilde H_{\tilde\mu} - \bar{\tilde\mu}\tilde H_{\bar{\tilde\mu}} = 0\,, \tag{4.51}$$

whence it follows again that

$$\tilde H(\tilde\mu,\bar{\tilde\mu},\lambda,\bar\lambda) = I(b)\,.$$

4.3 From the μ representation to the $(F, \bar{F})$ Lagrangian

In the μ representation, the basic algebraic equation (4.25) implies

$$\varphi = -H_{\mu}(1+\mu)^2\,, \quad \bar{\varphi} = -H_{\bar{\mu}}(1+\bar{\mu})^2\,. \tag{4.52}$$

It will be convenient to deal with the variable $\tilde{\mu}$ defined in (4.43). Keeping in mind that for the $\mathrm{Sp}(2,\mathbb{R})$ invariant case

$$H_{\tilde{\mu}} = I'(b)\bar{\tilde{\mu}}\,, \quad H_{\bar{\tilde{\mu}}} = I'(b)\tilde{\mu}\,, \quad I'(b) := \frac{dI}{db}\,,$$

we can rewrite (4.52) as

$$\varphi = -\frac{1}{1-\lambda\bar{\lambda}}\, I'\Big[\bar{\tilde{\mu}}(1+\lambda)^2 + 2b(1+\lambda)(1+\bar{\lambda}) + \tilde{\mu}b(1+\bar{\lambda})^2\Big], \quad \text{and c.c.}\,. \tag{4.53}$$

Redefining

$$\tilde{\mu} = \frac{1+\lambda}{1+\bar{\lambda}}\,\hat{\mu}\,, \quad \bar{\tilde{\mu}} = \frac{1+\bar{\lambda}}{1+\lambda}\,\bar{\hat{\mu}}\,, \quad b = \tilde{\mu}\bar{\tilde{\mu}} = \hat{\mu}\bar{\hat{\mu}} = \hat{b}\,, \tag{4.54}$$

where $\hat{\mu}$ and $\bar{\hat{\mu}}$ can be checked to transform just as in (3.26), and introducing

$$\hat{\varphi} := \frac{1-\lambda\bar{\lambda}}{(1+\lambda)(1+\bar{\lambda})}\,\varphi\,, \quad \bar{\hat{\varphi}} = \frac{1-\lambda\bar{\lambda}}{(1+\lambda)(1+\bar{\lambda})}\,\bar{\varphi}\,, \tag{4.55}$$

which, in the S_1, S_2 parametrization, coincide with the quantities defined in (2.13), we rewrite (4.53) as

$$\hat{\varphi} = -(\bar{\hat{\mu}} + 2\hat{b} + \hat{b}\hat{\mu})\, I'\,, \quad \bar{\hat{\varphi}} = -(\hat{\mu} + 2\hat{b} + \hat{b}\bar{\hat{\mu}})\, I'\,, \quad \text{and c.c.}\,, \tag{4.56}$$

or

$$\hat{\varphi} + \bar{\hat{\varphi}} + 4I'\hat{b} = -(\hat{\mu} + \bar{\hat{\mu}})\,(1+\hat{b})\, I'\,, \quad \hat{\varphi} - \bar{\hat{\varphi}} = (\hat{\mu} - \bar{\hat{\mu}})\,(1-\hat{b})\, I'\,. \tag{4.57}$$

Eqs. (4.56) are recognized as the basic equation (3.35) (and its conjugate) of the $\hat{b}$ representation of the NR formulation presented in the first part of the paper. So, already at this step we conclude that, on the shell of the auxiliary equation (4.25), the b representations of both formulations of the $\mathrm{Sp}(2,\mathbb{R})$ self-duality are the same, which implies that both these formulations yield the same eventual answer for the general $\mathrm{Sp}(2,\mathbb{R})$ self-dual Lagrangian $\tilde{L}^{\mathrm{sd}}(\lambda, F)$.

It seems instructive to consider here the consequences of eqs. (4.57) in more detail. These equations have the same form as the equations in the b representation for the U(1) case without scalars (eqs. (2.37) in [10]), with the only change $\varphi \to \hat{\varphi}\,, \bar{\varphi} \to \bar{\hat{\varphi}}$ and $\hat{\mu}, \bar{\hat{\mu}}$ instead of $\mu, \bar{\mu}$. Hence, as a corollary they imply the same algebraic equation (eq. (2.38) in [10])

$$(\hat{b}+1)^2\hat{\varphi}\bar{\hat{\varphi}} = \hat{b}\,[\hat{\varphi} + \bar{\hat{\varphi}} - (\hat{b}-1)^2 I']^2\,, \tag{4.58}$$

which expresses $\hat{b}$ in terms of the variables $\hat{\varphi}, \bar{\hat{\varphi}}$ defined in (4.55). Hence, the solution for $\hat{\mu}, \bar{\hat{\mu}}$ is obtained through the substitution $\varphi \to \hat{\varphi}\,, \bar{\varphi} \to \bar{\hat{\varphi}}$ in the solution for $\mu, \bar{\mu}$ of the case without scalar fields.

It remains to find the general expression for the Lagrangian in the original $(\varphi, \bar{\varphi})$ representation. The formulas one starts with mimic the U(1) case

$$\tilde{L}^{sd}(\lambda, \varphi, \bar{\varphi}) = \varphi\tilde{L}^{\mathrm{sd}}_{\varphi} + \bar{\varphi}\tilde{L}^{\mathrm{sd}}_{\bar{\varphi}} + I(\hat{b})\,, \quad \hat{b} = \hat{\mu}(\lambda, \varphi, \bar{\varphi})\bar{\hat{\mu}}(\lambda, \varphi, \bar{\varphi})\,, \tag{4.59}$$

where the λ dependence of the r.h.s. is hidden in the λ dependence of $\tilde{L}^{\rm sd}_\varphi, \tilde{L}^{\rm sd}_{\bar\varphi}$ and $\hat{b}$. The holomorphic derivatives $\tilde{L}^{\rm sd}_\varphi, \tilde{L}^{\rm sd}_{\bar\varphi}$ are related to the variables $\mu, \bar\mu$ also by the same relations as in the U(1) case

$$\tilde{L}^{\rm sd}_\varphi = \frac{\hat{E}_\nu - 1}{2(\hat{E}_\nu + 1)} = \frac{\mu - 1}{2(\mu + 1)}\,, \quad \text{and c.c.}\,, \tag{4.60}$$

which, with taking into account eqs. (4.52), yields

$$\tilde{L}^{\rm sd} = -\tfrac{1}{2}\, H_\mu\, (\mu^2 - 1) - \tfrac{1}{2}\, H_{\bar\mu}\, (\bar\mu^2 - 1) + I(b)\,. \tag{4.61}$$

The deviations from the pure U(1) case are revealed, when making use of the basic equations (4.57), (4.58) to eliminate μ and $\bar\mu$ in (4.61). After passing to the variables $\hat\mu, \bar{\hat\mu}$, we obtain

$$\tilde{L}^{\rm sd} = \frac{1}{2} I' \frac{1 - b}{1 - \lambda\bar\lambda}\Big[(1 - \lambda\bar\lambda)(\hat\mu + \bar{\hat\mu}) + (\lambda - \bar\lambda)(\hat\mu - \bar{\hat\mu})\Big] + I(\hat b)\,. \tag{4.62}$$

Using eqs. (4.57), we obtain the final expression for the Lagrangian

$$\tilde{L}^{\rm sd} = \frac{1}{2}\frac{\lambda - \bar\lambda}{(1 + \lambda)(1 + \bar\lambda)}(\varphi - \bar\varphi) + \hat{L}(\hat\varphi, \bar{\hat\varphi})\,, \tag{4.63}$$

$$\hat{L}(\hat\varphi, \bar{\hat\varphi}) = -\tfrac{1}{2}(\hat\varphi + \bar{\hat\varphi} + 4\hat{b} I')\,\frac{1 - \hat b}{1 + \hat b} + I(\hat b)\,. \tag{4.64}$$

The Lagrangian $\hat L$ accompanied by the algebraic equation (4.58) precisely yields the standard U(1) self-dual Lagrangian with the replacements $\varphi \to \hat\varphi, \bar\varphi \to \bar{\hat\varphi}$, where $\hat\varphi, \bar{\hat\varphi}$ are defined through $\varphi, \bar\varphi$ by eqs. (4.55) or (2.13). It satisfies the standard GZ self-duality constraint with respect to $\hat\varphi, \bar{\hat\varphi}$:

$$\hat\varphi - \bar{\hat\varphi} - 4\hat\varphi(\hat{L}_{\hat\varphi})^2 + 4\bar{\hat\varphi}(\hat{L}_{\bar{\hat\varphi}})^2 = 0\,. \tag{4.65}$$

The first term in (4.63) is the appropriate modification of the bilinear part of the action. This final answer for the nonlinear self-dual action is in the precise correspondence with the general action of Gibbons and Rasheed [2]. In the S_1, S_2 parametrization, using the relations (4.1), the Lagrangians (4.63) and (4.64) can be rewritten in the familiar form as

$$\tilde{L}^{\rm sd} = -\tfrac{i}{2} S_1(\varphi - \bar\varphi) + \hat{L}(\hat\varphi, \bar{\hat\varphi})\,, \tag{4.66}$$

$$\hat{L}(\hat\varphi, \bar{\hat\varphi}) = -\tfrac{1}{2}(\hat\varphi + \bar{\hat\varphi} + 4\hat{b} I')\,\frac{1 - \hat b}{1 + \hat b} + I(\hat b)\,. \tag{4.67}$$

Obviously, the Lagrangian (4.63) (or (4.66)) should be accompanied by the coset field Lagrangian $L'(\lambda) = L(S)$ given in (4.6). It is straightforward to check that the λ equations of motion following from the total Lagrangian $L^{\rm sd} = L'(\lambda) + \tilde{L}^{\rm sd}$ enjoy $\mathrm{Sp}(2,\mathbb{R})$ covariance.

4.4 Yet another derivation of the $\mathrm{Sp}(2,\mathbb{R})$ self-dual Lagrangian

Though the extended Lagrangian (4.26) contains a constrained auxiliary interaction $E(\nu, \bar\nu, \lambda, \bar\lambda)$ and we cannot immediately solve the relevant constraints (4.30) - (4.32), we know that it yields the correct description of the general $\mathrm{Sp}(2,\mathbb{R})$ self-dual systems, as was shown above by passing to its

μ representation. It turns out that the original E formulation is still capable to yield the general Lagrangian $L^{\rm sd}(\lambda,\varphi,\bar\varphi)$ of the $\mathrm{Sp}(2,\mathbb{R})$ systems without explicitly solving (4.30) - (4.32). These constraints amount to the triplet of alternative $\mathrm{Sp}(2,\mathbb{R})$ self-duality constraints on $\tilde L^{\rm sd}$ as they given, e.g., in [4].

Starting from the Lagrangian (4.26) and using the auxiliary equation (4.25) together with its corollaries

$$V\cdot F=\nu(1+\hat E_\nu)\,,\quad \varphi=\nu(1+\hat E_\nu)^2\,, \tag{4.68}$$

as well as the transformation properties (4.29), (4.34), (4.35) and

$$\delta\varphi=2i\gamma\varphi\,\frac{\hat E_\nu-1}{\hat E_\nu+1}+2\alpha\varphi\,\frac{1}{\hat E_\nu+1}+2\bar\alpha\varphi\,\frac{\hat E_\nu}{\hat E_\nu+1}\,, \tag{4.69}$$

we find the following simple $\mathrm{Sp}(2,\mathbb{R})$ transformation of the on-shell (i.e., with the algebraic equation (4.25) taken into account) Lagrangian:

$$\delta\tilde L^{\rm sd}=i\gamma[\varphi-\bar\varphi-2(\nu\hat E_\nu-\bar\nu\hat E_{\bar\nu})]+(\alpha-\bar\alpha)\,(\nu\hat E_\nu-\bar\nu\hat E_{\bar\nu})\,. \tag{4.70}$$

Defining as usual the general dual field strength

$$P_{\alpha\beta}=i\frac{\partial\tilde L^{\rm sd}}{\partial F^{\alpha\beta}}=i(F_{\alpha\beta}-2V_{\alpha\beta}) \tag{4.71}$$

and employing the auxiliary equation (4.25) together with its corollaries (4.68), it is easy to show that

$$F^{\alpha\beta}P_{\alpha\beta}-F^{\dot\alpha\dot\beta}P_{\dot\alpha\dot\beta}=i\nu(\hat E_\nu^2-1)-i\bar\nu(\hat E_{\bar\nu}^2-1)\,, \tag{4.72}$$

$$\varphi+P^2-\bar\varphi-\bar P^2=4(\nu\hat E_\nu-\bar\nu\hat E_{\bar\nu})\,. \tag{4.73}$$

On the other hand, the Lagrangian (4.26) can be rewritten on the shell of the auxiliary equation as

$$\tilde L^{\rm sd}=\tfrac12\nu(\hat E_\nu^2-1)+\tfrac12\bar\nu(\hat E_{\bar\nu}^2-1)+H\,,\quad H=\hat E-\nu\hat E_\nu-\bar\nu\hat E_{\bar\nu}\,. \tag{4.74}$$

Then, using (4.72), we can cast it in the standard form

$$\tilde L^{\rm sd}=\tfrac{i}{2}[(\bar F\cdot\bar P)-(F\cdot P)]+H\,. \tag{4.75}$$

It is straightforward to check that the variation (4.70) is entirely generated by the first term in (4.75), whence it follows that $\delta H=0$ in agreement with (4.36).

Substituting the expression (4.73) into the variation (4.70), we can rewrite it in a more standard way in terms of $\varphi,\bar\varphi,P^2$ and $\bar P^2$. This variation can be also used to find the $\mathrm{Sp}(2,\mathbb{R})$ analog of the standard GZ self-duality constraint in the $(F,\bar F)$ representation of the Lagrangian. To this end we can use the relations of the U(1) self-duality [9, 10]

$$\nu=\tfrac14\varphi(1-2\tilde L^{\rm sd}_\varphi)^2\,,\quad \hat E_\nu=\frac{1+2\tilde L^{\rm sd}_\varphi}{1-2\tilde L^{\rm sd}_\varphi}\,, \tag{4.76}$$

which are valid in the considered case too. We substitute these expressions into the r.h.s. of the variation (4.70) and alternatively calculate $\delta\tilde L^{\rm sd}$ as

$$\delta\tilde L^{\rm sd}=\delta\lambda\tilde L^{\rm sd}_\lambda+\delta\bar\lambda\tilde L^{\rm sd}_{\bar\lambda}+\delta\varphi\tilde L^{\rm sd}_\varphi+\delta\bar\varphi\tilde L^{\rm sd}_{\bar\varphi}\,, \tag{4.77}$$

where $\delta\varphi$ is defined in (4.69). With taking into account (4.76), the variation (4.69) can be rewritten as

$$\delta\varphi = 4i\gamma\,\tilde{L}^{\rm sd}_{\varphi} + \alpha\,\varphi(1 - 2\tilde{L}^{\rm sd}_{\varphi}) + \bar{\alpha}\,\varphi(1 + 2\tilde{L}^{\rm sd}_{\varphi})\,. \tag{4.78}$$

Substituting the explicit expressions for the variations of $\delta\lambda, \delta\varphi$ and their conjugates into $\delta\tilde{L}^{\rm sd}$ (4.77), and comparing the latter with (4.70), we obtain three conditions on the Lagrangian $\tilde{L}^{\rm sd}$ which are just the $\mathrm{Sp}(2,\mathbb{R})$ extension of the standard GZ constraint:

$$\begin{aligned}
&4(\lambda\tilde{L}^{\rm sd}_{\lambda} - \bar{\lambda}\tilde{L}^{\rm sd}_{\bar{\lambda}}) = \varphi[1 - 4(\tilde{L}^{\rm sd}_{\varphi})^2] - \bar{\varphi}[1 - 4(\tilde{L}^{\rm sd}_{\bar{\varphi}})^2]\,, &(4.79)\\
&4D_{\lambda}\tilde{L}^{\rm sd} = \varphi(1 - 2\tilde{L}^{\rm sd}_{\varphi})^2 - \bar{\varphi}(1 + 2\tilde{L}^{\rm sd}_{\bar{\varphi}})^2\,, &(4.80)\\
&4D_{\bar{\lambda}}\tilde{L}^{\rm sd} = \bar{\varphi}(1 - 2\tilde{L}^{\rm sd}_{\bar{\varphi}})^2 - \varphi(1 + 2\tilde{L}^{\rm sd}_{\varphi})^2\,, &(4.81)
\end{aligned}$$

where $D_{\lambda,\bar{\lambda}}$ were defined in (4.33). These constraints are equivalent to the sets (4.30) - (4.32) or (4.40), (4.41). One can explicitly check that the Lagrangian in the form (4.66), (4.67) solves eqs. (4.79) - (4.81). They can be cast in a more familiar form after passing to the coset representatives S_1, S_2 by the formulas (4.1). In the new parametrization, the λ derivatives are expressed as

$$\begin{aligned}
&\lambda\partial_{\lambda} - \bar{\lambda}\partial_{\bar{\lambda}} = \tfrac{i}{2}(\partial_S + \partial_{\bar{S}} + S^2\partial_S + \bar{S}^2\partial_{\bar{S}})\,,\\
&D_{\lambda} + D_{\bar{\lambda}} = -2(S\partial_S + \bar{S}\partial_{\bar{S}})\,,\\
&D_{\lambda} - D_{\bar{\lambda}} = i(\partial_S + \partial_{\bar{S}} - S^2\partial_S - \bar{S}^2\partial_{\bar{S}})\,.
\end{aligned} \tag{4.82}$$

Using these relations and going over to the tensorial notation:

$$\begin{aligned}
&\varphi\tilde{L}^{\rm sd}_{\varphi} + \bar{\varphi}\tilde{L}^{\rm sd}_{\bar{\varphi}} = \tfrac{1}{2}F^{mn}\frac{\partial\tilde{L}^{\rm sd}}{\partial F^{mn}}\,, \quad i(\varphi - \bar{\varphi}) = -\tfrac{1}{2}F^{mn}\tilde{F}_{mn}\\
&i[\varphi(\tilde{L}^{\rm sd}_{\varphi})^2 - \bar{\varphi}(\tilde{L}^{\rm sd}_{\bar{\varphi}})^2] = -\tfrac{i}{4}(P^2 - \bar{P}^2) = \tfrac{1}{8}G^{mn}\tilde{G}_{mn}\,,
\end{aligned} \tag{4.83}$$

we can bring (4.79) - (4.81) to the simple equivalent form given in [4]

$$\begin{aligned}
&2(S\tilde{L}^{\rm sd}_{S} + \bar{S}\tilde{L}^{\rm sd}_{\bar{S}}) = F^{mn}\frac{\partial\tilde{L}^{\rm sd}}{\partial F^{mn}}\,,\\
&\tilde{L}^{\rm sd}_{S} + \tilde{L}^{\rm sd}_{\bar{S}} = \tfrac{1}{4}F^{mn}\tilde{F}_{mn}\,,\\
&S^2\tilde{L}^{\rm sd}_{S} + \bar{S}^2\tilde{L}^{\rm sd}_{\bar{S}} = \tfrac{1}{4}G^{mn}\tilde{G}_{mn}\,.
\end{aligned} \tag{4.84}$$

The unique solution of these constraints is the general GR Lagrangian (1.1), (2.12) (with $L(S)$ subtracted)[3]. So the linear realization version of the bispinor auxiliary field formulation of the $\mathrm{Sp}(2,\mathbb{R})$ self-dual systems yields as the output the same GR Lagrangian as the formulation based on the nonlinearly transforming auxiliary fields, even without passing to the μ representation.

4.5 More on the interplay between the linear and nonlinear formulations

Here we give more details on the relationship between the auxiliary fields in the LR and NR formulations.

[3]Note that the constraints (4.79) - (4.81) or (4.84) are formulated for the Lagrangian $\tilde{L}^{\rm sd} = L^{\rm sd} - L(S)$, while the earlier employed the GZ-type constraints (2.14) or (2.23) only for its part $\hat{L}(\hat{\varphi}, \hat{\bar{\varphi}})$. These two sets of constraints are in fact equivalent to each other.

In Subsections 4.2 and 4.3 we observed that the Legendre transformation from the variables $\nu, \bar{\nu}$ to $\mu, \bar{\mu}$ performed in (4.26) yields in fact the same μ representation as the Legendre transformation from the variables $\hat{\nu}, \hat{\bar{\nu}}$ to $\hat{\mu}, \hat{\bar{\mu}}$ in the Lagrangian (3.16). It is interesting to reproduce the NR formulation, starting from the μ representation obtained in the framework of the LR formulation and applying just another type of the inverse Legendre transformation to this μ representation.

Namely, we start from the $\mathrm{Sp}(2,\mathbb{R})$ invariant function $H(\mu, \bar{\mu}, \lambda, \bar{\lambda})$ defined in (4.38), make the equivalency transformation from the variables $\mu, \bar{\mu}$ to $\hat{\mu}, \hat{\bar{\mu}}$ according to

$$\mu = \frac{(1+\lambda)\hat{\mu} + \lambda(1+\bar{\lambda})}{1+\bar{\lambda}+(1+\lambda)\bar{\lambda}\hat{\mu}}\,,$$

where the relations (4.43) and (4.54) were used, define

$$H(\mu, \bar{\mu}, \lambda, \bar{\lambda}) = \hat{H}(\hat{\mu}, \hat{\bar{\mu}}, \lambda, \bar{\lambda}) \tag{4.85}$$

and perform the Legendre transformation with respect to the variables $\hat{\mu}, \hat{\bar{\mu}}$:

$$\hat{H}_{\hat{\mu}} := -\hat{\nu}\,, \quad \hat{H}_{\hat{\bar{\mu}}} = -\hat{\bar{\nu}}\,, \qquad \hat{E} = \hat{H} - \hat{\mu}\hat{H}_{\hat{\mu}} - \hat{\bar{\mu}}\hat{H}_{\hat{\bar{\mu}}}\,. \tag{4.86}$$

Because $\delta\hat{H} = 0$, we immediately find

$$\delta\hat{\nu} = -(2i\gamma - \alpha + \bar{\alpha})\, S_2\, \hat{\nu} = -2i\rho\hat{\nu}\,, \quad \text{and c.c.}\,, \tag{4.87}$$

and

$$\delta\hat{E} = 0\,, \tag{4.88}$$

in the full agreement with the basic formulas of the NR representation. We also obtain

$$\hat{\mu} = \hat{E}_{\hat{\nu}}\,, \quad \hat{\bar{\mu}} = \hat{E}_{\hat{\bar{\nu}}}\,, \quad \hat{H} = \hat{E} - \hat{\nu}\hat{E}_{\hat{\nu}} - \hat{\bar{\nu}}\hat{E}_{\hat{\bar{\nu}}}\,. \tag{4.89}$$

The equation

$$\hat{\mu}\hat{H}_{\hat{\mu}} - \hat{\bar{\mu}}\hat{H}_{\hat{\bar{\mu}}} = 0 \tag{4.90}$$

implies

$$\hat{\nu}\hat{E}_{\hat{\nu}} - \hat{\bar{\nu}}\hat{H}_{\hat{\bar{\nu}}} = 0 \quad \Rightarrow \quad \hat{E} = \mathcal{E}(\hat{a})\,,\ \hat{a} = \hat{\nu}\hat{\bar{\nu}}\,. \tag{4.91}$$

Eq. (4.56) acquires the following form in terms of the newly introduced variables

$$\hat{\phi} = \hat{\nu}(1 + \hat{E}_{\hat{\nu}})^2\,. \tag{4.92}$$

Using the transformation laws of the variables $\hat{\mu}$, $\hat{\nu}$ and $\hat{\phi}$ it is straightforward to check the $\mathrm{Sp}(2,\mathbb{R})$ covariance of (4.92).

In this way the basic formulas of the NR formulation are recovered.

As for the Lagrangian (3.16), it can be restored from the requirement that it reproduces the basic relations of the NR formulation, including eq. (4.92). We introduce the bispinor field $\hat{V}_{\alpha\beta}, \hat{\bar{V}}_{\dot{\alpha}\dot{\beta}}$, such that

$$\hat{\nu} = \hat{V}^{\alpha\beta}\hat{V}_{\alpha\beta}\,, \quad \hat{\bar{\nu}} = \hat{\bar{V}}^{\dot{\alpha}\dot{\beta}}\hat{\bar{V}}_{\dot{\alpha}\dot{\beta}}\,, \tag{4.93}$$

and represent the sought Lagrangian as

$$\mathcal{L}(\lambda, F, \hat{V}) = \mathcal{L}_1(\lambda, \varphi) - 2\sqrt{S_2}[(\hat{V} \cdot F) + (\hat{\bar{V}} \cdot \bar{F})] + \hat{\nu} + \hat{\bar{\nu}} + \mathcal{E}(\hat{\nu}\hat{\bar{\nu}}) \,. \tag{4.94}$$

Varying it with respect to $\hat{V}_{\alpha\beta}$ gives

$$\sqrt{S_2}\, F_{\alpha\beta} = (1 + \hat{E}_{\hat{\nu}})\, \hat{V}_{\alpha\beta} \,, \tag{4.95}$$

which implies just (4.92). We also need the correct dynamical equations for $F_{\alpha\beta}$ on the shell of the algebraic constraint. In other words, we require

$$F_{\alpha\beta} - 2V_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial F^{\alpha\beta}} = \left(\frac{\partial \mathcal{L}_1}{\partial F^{\alpha\beta}} - 2\sqrt{S_2}\, \hat{V}_{\alpha\beta} \right), \tag{4.96}$$

with $\hat{V}_{\alpha\beta}$ being subjected to (4.95).

As the first step, we rewrite $F_{\alpha\beta} - 2V_{\alpha\beta}$ in terms of the variables $\hat{\nu}$

$$F_{\alpha\beta} - 2V_{\alpha\beta} = \Big(- iS_1 + S_2 \frac{\hat{E}_{\hat{\nu}} - 1}{\hat{E}_{\hat{\nu}} + 1} \Big) F_{\alpha\beta}. \tag{4.97}$$

Next we express $\hat{V}_{\alpha\beta}$ in (4.96) through $F_{\alpha\beta}$ by (4.95) and compare two expressions for $F_{\alpha\beta} - 2V_{\alpha\beta}$, which yields

$$\mathcal{L}_1 = \tfrac{1}{2} S_2\, (\varphi + \bar{\varphi}) - \tfrac{i}{2} S_1\, (\varphi - \bar{\varphi}) \,. \tag{4.98}$$

Finally, the Lagrangian (4.94) takes the form of (3.16) with the subtracted $L(S)$. Note that the auxiliary equation (4.95) together with (4.97) yield the same on-shell relations (3.24) ad (3.25) between the LR and NR tensorial fields.

The off-shell Lagrangians (4.26) and (3.16) are essentially different and seem not to be related to each other by any obvious field redefinition, though they yield the same system on the shell of the auxiliary equation.

5 Conclusions

We investigated $\mathrm{Sp}(2, \mathbb{R})$ duality-invariant interactions of scalar and electromagnetic fields, employing two different formulations involving auxiliary bispinors and/or auxiliary scalars ("μ representation"). The main emphasis was on the transformation properties of the relevant Lagrangians and their equations of motion. The formalism of Section 3 started from the nonlinear realization of $\mathrm{Sp}(2, \mathbb{R})$ on the basic auxiliary bispinor fields, while in Section 4 those auxiliary fields were taken to transform linearly. Both formalisms admit a Legendre-type transformation to Lagrangians with auxiliary scalar fields. This allowed us to prove that both auxiliary-field formulations yield equivalent self-dual Lagrangians in the standard (S, F) representation. Like in the U(1) duality case, any choice of a Lagrangian exhibiting $\mathrm{Sp}(2, \mathbb{R})$ duality amounts to a particular choice of an $\mathrm{Sp}(2, \mathbb{R})$ invariant unconstrained interaction of the auxiliary bispinor fields. The (S, F) Lagrangian emerges from the extended Lagrangian upon elimination of the auxiliary fields through their equations of motion in terms of the coset scalars S and the electromagnetic field strengths F.

It is rather straightforward to generalize our auxiliary-field formulations to the case of $\mathrm{Sp}(2N, \mathbb{R})$ duality as proper extensions of the analogous formulations for $\mathrm{U}(N)$ duality [11]. We hope to address this task elsewhere. The formalism of auxiliary superfields in $\mathcal{N} = 1$ supersymmetric self-dual theories [12, 13] can also be generalized, with additional chiral coset multiplets, to the case of noncompact dualities. We are curious to learn which of the two (if not both) approaches presented here admit an unambiguous extension to supersymmetric theories. One might also try to construct an $\mathrm{Sp}(2,\mathbb{R})$ (and $\mathrm{Sp}(2N,\mathbb{R})$) version of the "hybrid" formulation of U(1) duality [19], which joins the auxiliary-tensor approach with the manifestly Lorentz- and duality-invariant PST formalism (see [20] and references therein).

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